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MATHEMATICS

Stabilization of the Solution to the Identification Problem of the Source Function for a One-Dimensional Parabolic Equation

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In [2, 7], global unique solvability and stabilization were proved for the solution to the problem of identifying the source function of a parabolic equation with Cauchy data when the time variable tends to infinity and the initial data of the problem tend sufficiently rapidly to zero with respect to a distinguished spatial variable.

We find sufficient conditions for the global solvability and stabilization of the solution to this identification problem in classes of smooth bounded functions. The Cauchy problem and boundary value problems with Dirichlet and Neumann conditions are studied.

In the domain $\Pi_{(0,T)} = \{(t, x)| \, 0 < t < T, x \in E_1\}$, we consider the Cauchy problem

$$u_t(t, x) = b(t)u_{xx}(t, x) + a(t)u(t, x) + f(t)g(t, x), \quad (1)$$

$$u(0, x) = u_0(x), \quad x \in E_1. \quad (2)$$

Here, $a(t)$, $b(t)$, $u_0(x)$, and $g(t, x)$ are real-valued functions defined in $[0, T]$, E_1, and $\Pi_{[0,T]}$, respectively. In what follows, we assume that $b(t) > 0$ for all t in the domain under consideration.

The function $f(t)$ has to be determined together with a solution $u(t, x)$ to problem (1), (2) that satisfies the overdetermination condition

$$u(t, 0) = \varphi(t), \quad t \in [0, T], \quad (3)$$

under the compatibility condition

$$u_0(0) = \varphi(0). \quad (4)$$

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Let $g(t, x)$ be such that. for $t \in [0, T]$,

$$|g(t, 0)| \geq \delta > 0, \quad (5)$$

where δ is a fixed constant.

The initial data are assumed to be sufficiently smooth, have all the continuous derivatives involved in the following relation, and satisfy it for $(t, x) \in \Pi_{[0,T]}$:

$$|a(t)| + |b(t)| + |\varphi(t)| + |\varphi'(t)| + \left|\frac{d^k}{dx^x}u_0(x)\right| + \left|\frac{\partial^k}{\partial x^x}g_0(t, x)\right| \leq C, \quad (6)$$
$$k = 0, 1, 2, 3, 4.$$

In view of condition (3), the unknown coefficient can be expressed as

$$f(t) = \frac{\gamma(t) - b(t)u_{xx}(t, 0)}{g(t, 0)}, \quad (7)$$

where $\gamma(t) = \varphi'(t) - a(t)\varphi(t)$ is a given smooth function.

To prove solvability, we use the weak approximation method [1, 4].

The following theorems hold.

Theorem 2. *Let conditions (4)–(6) be satisfied. Then problem (1)–(3) has a unique solution* $u(t, x), f(t)$ *in the class*

$$Z_{[0,T]} = \{u(t, x), f(t)| \, u(t, x) \in C^{1,2}_{t,x}(\Pi_{[0,T]}), \; f(t) \in C[0, T]\},$$

and that solution satisfies the relation

$$|f(t)| + \sum_{k=0}^{2}\left|\frac{\partial^k}{\partial x^k}u(t, x)\right| \leq C, \quad (t, x) \in \Pi_{[0,T]}, \quad (8)$$

where

$$C^{1,2}_{t,x}(\Pi_{[0,T]}) = \{u(t,x)|\ u, u_t, u_x, u_{xx} \in C(\Pi_{[0,T]})\}.$$

We analyze the behavior of the solution to problem (1)–(3) in $\Pi_{[0,+\infty)}$ as the time variable tends to infinity.

Theorem 2. *Let, in* $\Pi_{[0,+\infty)}$, *conditions (5) and (6) be satisfied and*

$$a(t) \le -\tilde{A},\ \ \text{где}\ \ \tilde{A} > \sup_{\eta\in[0,+\infty)}\sup_{x\in E_1}\left|\frac{b(\eta)g_{xx}(\eta,x)}{g(\eta,0)}\right|,$$
$$\int_0^{+\infty}|b(\eta)|d\eta + \int_0^{+\infty}|\varphi'(t)-\varphi(t)a(t)|d\eta < C. \tag{9}$$

Then the solution to problem (1)–(3) in $\Pi_{[0,+\infty)}$ *satisfies the inequality*

$$|f(t)| + |u(t,x)| \le C.$$

If, additionally,

$$|b(t)| \le \frac{Q_1}{1+t^p},\quad p = \text{const} > 1,\quad Q_1 = \text{const} > 0, \tag{10}$$

$$|\varphi'(t) - a(t)\varphi(t)| \le \frac{Q_2}{1+t^q},\quad q = \text{const} > 1,\quad Q_2 = \text{const} > 0, \tag{11}$$

then the solution to problem (1)–(3) in $\Pi_{[0,+\infty)}$ *satisfies the relation*

$$\lim_{t\to+\infty}\left(\sup_{x\in E_1}|u(t,x)| + |f(t)|\right) = 0.$$

Similar results hold for the Dirichlet and Neumann boundary value problems with homogeneous boundary conditions.

In the domain $\Omega_{[0,T]} = \{(t,x)|\ 0 \le t \le T,\ 0 \le x \le \pi\}$, consider the Dirichlet boundary value problem for Eq. (1) with initial conditions (2) and the boundary conditions

$$u(t,0) = u(t,\pi) = 0. \tag{12}$$

The function $f(t)$ has to be determined simultaneously with a solution $u(t,x)$ to problem (1), (2), (12) that satisfies the overdetermination condition

$$u(t,\gamma) = \varphi(t) \tag{13}$$

for some fixed point $0 < \gamma < \pi$.

Assume that the matching condition

$$u_0(\gamma) = \varphi(0) \tag{14}$$

holds and, for $0 \le t < \infty$,

$$|g(t,\gamma)| \ge \delta > 0,\quad \delta = \text{const}. \tag{15}$$

Suppose that $u_0(x)$ and $g(t,x)$ can be extended to E_1 as odd functions of x (with the notation retained):

$$u_0(x) = \sum_{k=0}^{\infty}\alpha_k \sin kx,\quad g(t,x) = \sum_{k=0}^{\infty}\beta_k(t)\sin kx, \tag{16}$$

where α_k = const and $\beta_k(t) \in C[0,T]$. Moreover, let these functions be sufficiently smooth (have all the continuous derivatives involved in (6)) and condition (6) be satisfied for $(t,x) \in \Pi_{[0,T]} = \{(t,x)|\ 0 \le t \le T, x \in E_1\}$.

Theorem 3. *Let conditions (6) and (14)–(16) be fulfilled. Then, for any T > 0 in the class*

$$\hat{Z}(T) = \{u(t,x), f(t)|\ u \in C^{1,2}_{t,x}(\Omega_{[0,T]}),\ f(t) \in C[0,T]\}$$

problem (1), (2), (12), (13) has a unique solution $u(t,x)$, $f(t)$ *that satisfies*

$$|f(t)| + \sum_{k=0}^{2}\left|\frac{\partial^k}{\partial x^k}u(t,x)\right| \le C,\quad (t,x) \in \Omega_{[0,T]}. \tag{17}$$

Theorem 4. *Let in* $\Omega_{[0,+\infty)}$ *conditions (6) and (15) be fulfilled and*

$$a(t) \le -\tilde{A},\quad \text{where}\ \ \tilde{A} > \sup_{\eta\in[0,+\infty)}\sup_{x\in[0,\pi]}\left|\frac{b(\eta)g_{xx}(\eta,x)}{g(\eta,0)}\right|, \tag{18}$$

$$\int_0^{+\infty}|b(\eta)|d\eta + \int_0^{+\infty}|\varphi'(t)-\varphi(t)a(t)|d\eta \le C. \tag{19}$$

Then the solution to problem (1), (2), (12), (13) in $\Omega_{[0,+\infty)}$ *satisfies the inequality*

$$|f(t)| + |u(t,x)| \le C. \tag{20}$$

If inequalities (10) and (11) additionally hold, then the solution $u(t,x)$, $f(t)$ *to problem (1), (2), (12), (13) satisfies*

$$\lim_{t\to+\infty}\left(\sup_{x\in[0,\pi]}|u(t,x)| + |f(t)|\right) = 0. \tag{21}$$

Similar theorems hold for the Neumann boundary value problem if $u_0(x)$ and $g(t,x)$ are extended to E_1 as even functions of x.

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